

# $B_K$ from the lattice with Wilson quarks

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**Abstract.** We report our results for the bag parameter  $B_K$  obtained from the quenched simulations on the lattice with Wilson fermions for three values of the lattice spacing. We implemented the method by which no subtraction of the mixing with other four-fermion  $\Delta S = 2$  operators is needed. Our final result, in terms of the renormalization group invariant bag parameter, is  $\hat{B}_K = 0.96 \pm 0.10$ .

## 1 Introduction

The parameter characterizing the size of the indirect  $CP$ -violation in the system of neutral kaons,  $\varepsilon_K$ , has been accurately measured since long ago [1]. The precise theoretical estimate of the corresponding  $K^0-\bar{K}^0$  mixing amplitude, however, is still missing due to uncertainties in the computation of the matrix element of the operator

$$\begin{aligned} O^{\Delta S=2} &= (\bar{s}^A \gamma_\mu (1 - \gamma_5) d^A) (\bar{s}^B \gamma_\mu (1 - \gamma_5) d^B) \\ &= Q_1 + \mathcal{Q}_1, \end{aligned} \quad (1)$$

where  $Q_1$  and  $\mathcal{Q}_1$  are respectively the parity conserving and parity violating part of  $O^{\Delta S=2}$ .  $A$  and  $B$  are the color indices. The matrix element of the renormalized operator,

$$\begin{aligned} \langle \bar{K}^0 | \hat{O}^{\Delta S=2}(\mu) | K^0 \rangle &= \langle \bar{K}^0 | \hat{Q}_1(\mu) | K^0 \rangle \\ &= \frac{8}{3} f_K^2 m_K^2 B_K(\mu), \end{aligned} \quad (2)$$

is conveniently parameterized in terms of the bag parameter  $B_K$ , the measure of the deviation of the matrix element from its value obtained in the vacuum saturation approximation (in which  $B_K = 1$ ).

Over the past two decades quite an impressive progress in computing  $B_K$  on the lattice has been made. We now know how to renormalize the four-fermion operator  $Q_1$  non-perturbatively in the RI/MOM [2] and in the Schrödinger functional scheme [3]. We also know how to relate to  $\hat{B}_K$ , the renormalization scheme invariant bag parameter, since the anomalous dimension is calculated in a number of schemes at NLO in continuum perturbation theory [4–6], the same accuracy at which the corresponding Wilson coefficient has been calculated [7]. A high statistics computation of  $B_K$  with Wilson fermions for several values of the lattice spacing  $a$  has been performed in [8]. Preliminary unquenched calculations have

been made too [9]. However, all the works in which the Wilson quarks were used suffer from the potentially large systematic uncertainty that arises from the large mixing of the operator  $Q_1$  with other parity-even operators  $Q_{2-5}$  which have different naive chiralities<sup>1</sup>. That feature is a consequence of the explicit chiral symmetry breaking induced by the Wilson term in the quark action. In other words, the renormalization pattern of the lattice operator  $Q_1$  regularized à la Wilson is

$$\hat{Q}_1(\mu) = Z(a\mu) \left[ Q_1(a) + \sum_{i=2}^5 \Delta_i(a) Q_i(a) \right], \quad (3)$$

where  $Z(a\mu)$  is the multiplicative renormalization constant present also in formulations where chiral symmetry is preserved, while  $\Delta_{2-5}(a)$  are the mixing coefficients peculiar for the Wilson regularization. The difficulty is not only that one needs to compute the subtraction constants  $\Delta_{2-5}(a)$  non-perturbatively but one should also compute them very accurately because the lattice regularized bare matrix elements  $\langle Q_{2-5} \rangle$  are orders of magnitude larger than  $\langle Q_1 \rangle$ . Therefore, even though the subtraction constants are numerically very small (see the tables in [10]<sup>2</sup>), the net effect of the subtractions is large. It is clearly desirable to have a method that allows one to compute the matrix element (2) without necessity to subtract the mixing.

In this article we use the hadronic Ward identity, proposed in [11], to relate the matrix element of the operator  $Q_1$  to the parity violating one,  $\mathcal{Q}_1$ . The latter does not suf-

<sup>1</sup> For the explicit forms of all the parity-even operators,  $Q_{1-5}$ , see e.g. Sect. 3.2 of [2].

<sup>2</sup> To avoid a notational ambiguity we point out that the subtraction constants  $\Delta_{12}(a), \dots, \Delta_{15}(a)$  presented in [10] correspond to  $\Delta_2(a), \dots, \Delta_5(a)$  used in this article.

fer from the spurious mixing and thus the problem of mixing with other dimension-six operators is circumvented. The price to pay is that one has to compute a four-point correlation function where one pion is integrated over all lattice space-time coordinates. Similar in spirit, but quite different in practice, is the proposal made in [12] where the chiral rotation has been added to the mass term so as to kill the spurious lattice mixings. A preliminary study of the  $B_K$  parameter by using that method has been presented in [13].

In Sect. 2 we will briefly recall the basic elements of the Ward identity method to compute  $B_K$  without subtractions; in Sect. 3 we present the results for the matrix element (2) for the directly accessible pseudoscalar meson masses from which we will extract the  $\hat{B}_K$  parameter; in Sect. 4 we briefly draw our conclusions.

## 2 Strategy

In this section we will briefly recall the main steps involved in the extraction of the  $B_K$ -parameter by using both methods, with and without subtractions.

### 2.1 Standard procedure: “with subtractions”

The standard way to extract the matrix element (2) proceeds through the computation of the correlation functions

$$\begin{aligned} G_{K_P^0}(t) &= \langle K_P^0 \dagger(t) K_P^0(0) \rangle, \\ G_{\hat{Q}_1}(t_x, t_y) &= \langle K_P^0 \dagger(t_x) \hat{Q}_1(0) K_P^0(t_y) \rangle, \end{aligned} \quad (4)$$

with  $\hat{Q}_1$  defined in (3), and  $K_P^0(t_x) = \sum_{\mathbf{x}} \bar{d}(x) \gamma_5 s(x)$ . Therefore to get  $G_{\hat{Q}_1}(t_x, t_y)$  one must compute the correlators by using the complete set of parity conserving four-fermion operators,  $Q_{1-5}$ , subtract the spurious mixing, and provide the multiplicative renormalization, as indicated in (3). This procedure is particularly delicate because the approximate restoration of chiral symmetry (which is exactly recovered only in the continuum limit) depends on how well the subtractions are made. The subtraction constants  $\Delta_{2-5}$  do not depend on the renormalization scheme. Their values have been recently estimated non-perturbatively, in the RI/MOM scheme [10].

The matrix element (2) is extracted from the study of the large time asymptotic behavior of the ratio

$$\begin{aligned} R^{\text{stand}}(t_y) &= \frac{G_{\hat{Q}_1}(t_x, t_y)}{G_{K_P^0}(t_x) G_{K_P^0}(t_y)} \quad \Rightarrow \quad \frac{\langle \bar{K}^0 | \hat{Q}_1 | K^0 \rangle}{\langle 0 | K_P^0 | K^0 \rangle^2}, \end{aligned} \quad (5)$$

where we fix one of the source operators at  $t_x$  so that the kaon state which is created by the four-fermion operator in the origin is already asymptotic when annihilated by  $K_P^0(t_x)$ . The time  $t_y$ , instead, is free. On the plateaus,  $T \gg t_y \gg T/2$ , where all the operators are far away from one another, we read off the desired matrix element divided by the pseudoscalar density squared.

### 2.2 Alternative procedure: “without subtractions”

The method proposed in [11] is based on the use of a Ward identity which arises from applying the  $\tau_3$  axial rotation,

$$\begin{aligned} \delta u(x) &= i\alpha(x) \gamma_5 u(x), & \delta \bar{u}(x) &= i\alpha(x) \bar{u}(x) \gamma_5, \\ \delta d(x) &= -i\alpha(x) \gamma_5 d(x), & \delta \bar{d}(x) &= -i\alpha(x) \bar{d}(x) \gamma_5, \end{aligned} \quad (6)$$

onto the matrix element  $\langle \hat{K}_P^0(x) \hat{Q}_1(0) \hat{K}_P^0(y) \rangle$ , where  $\hat{K}_P^0 = Z_P K_P^0$ . To write down the relevant Ward identity, we introduce the bilinear operators

$$\begin{aligned} K_S^0(t) &= \sum_{\mathbf{x}} \bar{d}(x) s(x), \\ \Pi^0(x) &= \bar{d}(x) \gamma_5 d(x) - \bar{u}(x) \gamma_5 u(x), \end{aligned} \quad (7)$$

and the corresponding renormalized  $\hat{K}_S^0(t) = Z_S K_S^0(t)$ . With these definitions at hand the renormalized Ward identity reads

$$\begin{aligned} &2 \langle \hat{K}_P^0(t_x) \hat{Q}_1(0) \hat{K}_P^0(t_y) \rangle \\ &= 2m \sum_z \langle \hat{\Pi}^0(z) \hat{K}_P^0(t_x) \hat{Q}_1(0) \hat{K}_P^0(t_y) \rangle \\ &\quad - \langle \hat{K}_S^0(t_x) \hat{Q}_1(0) \hat{K}_P^0(t_y) \rangle - \langle \hat{K}_P^0(t_x) \hat{Q}_1(0) \hat{K}_S^0(t_y) \rangle \\ &\quad + \mathcal{O}(a), \end{aligned} \quad (8)$$

where, in view of the fact that we work in the chiral limit, we dropped the sum over the space-time of the term containing the divergence of the axial current, which appears, together with the first term on the RHS containing  $2m\Pi^0(z)$ . The dropped term is zero when there is no momentum transfer between the initial and final states, provided that no singularities are encountered. Working at non-zero quark mass then ensures that infrared divergences are avoided, so that the integration over space-time indeed yields a vanishing value. In practice, we work in the SU(3) limit, i.e., we take all three quarks to be degenerate in mass,  $m_u = m_d = m_s \equiv m$ . The term on the LHS of (8), corresponding to the rotation of the operator  $\hat{Q}_1$ , is the desired matrix element. The last two terms in (8) correspond to the rotation of the pseudoscalar kaon sources. These terms, although necessary to saturate the Ward identity, disappear in the SU(3) limit as shown in the appendix. Thus, the Ward identity we use in practice reads

$$\begin{aligned} &\langle K_P^0(t_x) \hat{Q}_1(0) K_P^0(t_y) \rangle \\ &= m(a\mu) \mathcal{Z}(a\mu) \sum_z \langle \hat{\Pi}^0(z) K_P^0(t_x) \mathcal{Q}_1(0) K_P^0(t_y) \rangle \\ &\equiv G_{\mathcal{Q}_1}(t_x, t_y), \end{aligned} \quad (9)$$

where we stress the presence of  $\mathcal{O}(a)$  artefacts, i.e., the four-fermion operators are not improved. Owing to CPS symmetry the parity-odd operator  $\mathcal{Q}_1$  renormalizes multiplicatively only.  $\mathcal{Z}(a\mu)$  has been recently computed non-perturbatively in the RI/MOM scheme [10]. We use the quark mass,  $m(a\mu) = \rho Z_A(a) / Z_P(a\mu)$ , defined through

the non-singlet axial Ward identity,

$$\rho = \frac{\langle \nabla_0 A_0(t) K_P^{0\dagger}(0) \rangle}{2 \langle K_P^0(t) K_P^{0\dagger}(0) \rangle}, \quad (10)$$

where  $A_\mu(t) = \sum_{\mathbf{x}} \bar{s}(x) \gamma_\mu \gamma_5 d(x)$ , and  $Z_A(a)$  is the axial-current renormalization factor [10]. Notice also that in (9) the renormalization constant of the pseudoscalar density,  $Z_P(a\mu)$ , in  $m(a\mu)$  cancels against the one in  $\hat{\Pi}^0(z)$ . In terms of Feynman diagrams, (9) can be written as

$$\begin{aligned} & 2 \left[ C8(t_x, t_y) + D8(t_x, t_y) \right] \\ &= 2Z_A \rho \left[ CE(t_x, t_y) + CE(t_y, t_x) + DE(t_x, t_y) \right. \\ & \quad \left. + DE(t_y, t_x) \right], \end{aligned} \quad (11)$$

where  $C8(t_x, t_y)$  and  $D8(t_x, t_y)$  correspond to the connected and disconnected “eight” diagrams, while  $CE(t_y, t_x)$  and  $DE(t_y, t_x)$  refer to the connected and disconnected “emission” diagrams shown in Fig.1 of [11]. Proceeding like in the standard method, the matrix element is extracted from the study of the ratio,

$$\begin{aligned} & R^{\text{w/o subtr.}}(t_y) \\ &= \frac{G_{Q_1}(t_x, t_y)}{G_{K_P^0}(t_x) G_{K_P^0}(t_y)} \Big|_{\tau \gg t_y \gg \tau/2} \Rightarrow \frac{\langle \bar{K}^0 | \hat{Q}_1 | K^0 \rangle}{|\langle 0 | K_P^0 | K^0 \rangle|^2}. \end{aligned} \quad (12)$$

### 3 Extraction of $B_K$

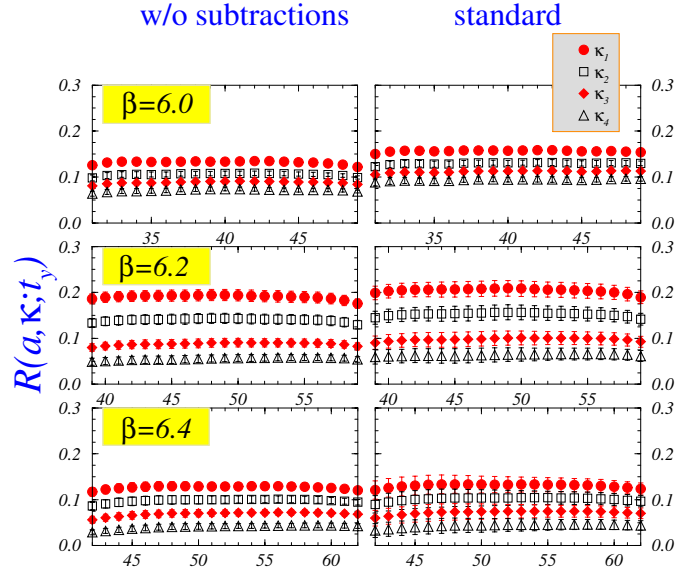
In this section we present our main results. We use both procedures, the standard and the one without subtractions, which provides us a useful cross-check. Of course the two methods suffer from  $\mathcal{O}(a)$  effects that are different in size, but should converge to the same value in the continuum limit.

#### 3.1 Lattices and signals for the ratios (5) and (12)

The details of our lattice setups were presented in our previous publications [10,14]. We work at three lattice spacings which correspond to  $\beta = 6.0, 6.2$ , and to  $6.4$ . In each simulation we work with four different values of the quark mass, i.e., with four values of the parameter  $\kappa$ , and compute the correlation functions needed to form the ratios (5) and (12).

In Fig. 1 we show the signals we obtain by using both methods and for all quark masses used in our simulations (the  $\kappa$  are ordered as  $m_1 > m_2 > m_3 > m_4$ ). The plateaus correspond to the signals for the bare operators, i.e. without accounting for the overall (scale dependent) renormalization constants  $Z(a\mu)$  and  $\mathcal{Z}(a\mu)$ . For the standard method we need to specify the subtraction constants  $\Delta_{2-5}(a)$ . We use the results recently obtained in [10]. In our calculation one source operator is fixed at

$$t_x = 12|_{\beta=6.0}, 15|_{\beta=6.2}, 17|_{\beta=6.4}, \quad (13)$$



**Fig. 1.** Plateaus for the ratios defined in (5) and (12) which are referred to as the standard procedure (right plots) and the one without subtractions (left plots). Plateaus are displayed for all 4 quark masses and for all three  $\beta$ 's used in this work

after having checked that the signal does not change for larger  $t_x$ , except that the plateaus become slightly shorter.

To account for the multiplicative renormalization we proceed as follows. We employ the method described in detail in [10], to compute the renormalization constants  $Z(a\mu)$  and  $\mathcal{Z}(a\mu)$  in the RI/MOM scheme at about 20 different values of the scale  $a\mu$ . We then convert such renormalized ratios  $R^{\text{stand.}}(t_y)$  and  $R^{\text{w/o subtr.}}(t_y)$  into their renormalization invariant forms by using the perturbative anomalous dimension known at NLO accuracy [5], namely

$$\begin{aligned} \langle Q_1 \rangle^{\text{rgi}} &= \alpha_s(\mu)^{-\gamma_0/2\beta_0} \left( 1 + \frac{\alpha_s(\mu)}{4\pi} J_{\text{RI/MOM}} \right) \\ & \times \langle Q_1(\mu) \rangle^{\text{RI/MOM}}, \end{aligned} \quad (14)$$

where  $\gamma_0 = 4$  and  $\beta_0 = 11 - 2n_f/3$  are universal and

$$J_{\text{RI/MOM}} = 8 \log 2 - \frac{17397 - 2070n_f + 104n_f^2}{6(33 - 2n_f)^2} \quad (15)$$

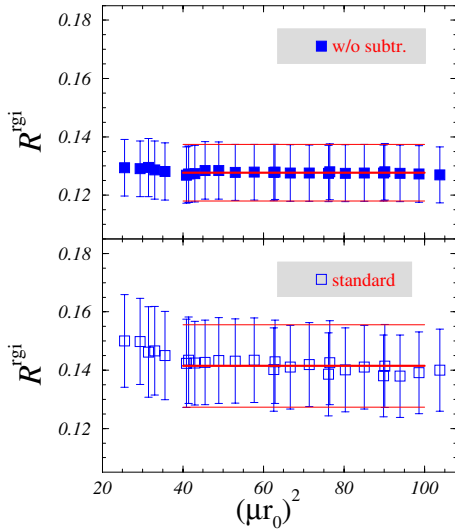
The plateaus used to fit the ratios  $R^{\text{stand.}}(t_y)$  and  $R^{\text{w/o subtr.}}(t_y)$  to constants  $R^{\text{stand.}}$  and  $R^{\text{w/o subtr.}}$ , respectively, for each value of the renormalization scale  $a\mu$ , are

$$t_{\text{plateau}} \in \{ [36, 42]_{\beta=6.0}, [43, 52]_{\beta=6.2}, [49, 58]_{\beta=6.4} \}. \quad (16)$$

In Fig. 2 we illustrate both ratios computed at 24 different values of the renormalization scale at  $\beta = 6.2$ , and then converted to the renormalization group invariant form. After proceeding similarly for the other lattice spacings, we find that for  $(r_0\mu)^2 \geq 40$ , the ratios  $\hat{R}^{\text{stand.}}$  and  $\hat{R}^{\text{w/o subtr.}}$  nicely follow the perturbatively established scale dependence (14). After fitting these results to a constant on the interval  $40 \leq (r_0\mu)^2 \leq 100$  we obtain the results listed in

**Table 1.** The values of the (bare) quark mass,  $\rho$ , obtained by using the axial Ward identity (10), of the quantity  $X$ , defined in (17), and of the matrix element  $R$  obtained by using both methods. The  $R$  are computed from the fit of the ratios (5) and (12) to a constant on the plateau intervals indicated in (16)

$\beta$	$\kappa$	$\rho \times 10^2$	$X(m_q)$	$\hat{R}^{\text{stand.}}(a)$	$\hat{R}^{\text{w/o subtr.}}(a)$
6.0	0.1335	5.997(7)	0.1256(20)	0.1356(59)	0.1152(45)
	0.1338	4.368(8)	0.1051(19)	0.1124(55)	0.0925(42)
	0.1340	3.750(8)	0.0910(18)	0.0969(52)	0.0776(39)
	0.1342	3.129(9)	0.0765(16)	0.0812(48)	0.0629(36)
6.2	0.1339	5.792(7)	0.1766(53)	0.188(13)	0.1732(97)
	0.1344	4.268(7)	0.1348(48)	0.142(12)	0.1277(86)
	0.1349	2.748(7)	0.0891(40)	0.092(12)	0.0802(72)
	0.1352	1.834(8)	0.0589(35)	0.060(11)	0.0503(57)
6.4	0.1347	3.144(2)	0.1336(45)	0.1363(93)	0.1226(79)
	0.1349	2.540(3)	0.1078(45)	0.1087(88)	0.0954(74)
	0.1351	1.937(3)	0.0809(44)	0.0804(79)	0.0681(65)
	0.1353	1.334(3)	0.0536(41)	0.0518(65)	0.0409(49)



**Fig. 2.** Fit of the ratio  $R(t_y)$  [see (5) and (12)] to a constant  $\hat{R} \equiv R^{\text{rgi}}$  in the interval  $40 \leq (\mu r_0)^2 \leq 100$ . The lower plot corresponds to the case in which the effect of mixing with other  $\Delta S = 2$  operators has been subtracted. In the upper plot the Ward identity method (without subtractions) has been employed. Illustration is provided with the results collected at  $\beta = 6.2$ , and for  $\kappa = 0.1344$

Table 1. To convert from  $a\mu$  to  $r_0\mu$  we used the accurately estimated  $r_0/a$  from [15], while in the calculation of the two-loop running coupling,  $\alpha_s(r_0\mu)$ , we used the quenched value  $r_0\Lambda_{\overline{\text{MS}}}^{(n_f=0)} = 0.602(48)$  [16]. In Table 1 we also give the values of the bare quark mass  $\rho$ , computed from the axial Ward identity (10), and of the quantity  $X(m_q)$  defined as

$$\frac{8}{3} \frac{Z_A^2 \langle A_0(t) A_0^\dagger(0) \rangle}{\langle K_P^0(t) K_P^{0\dagger}(0) \rangle} \Big|_{\tau \gg t \gg 0} \Rightarrow \frac{8}{3} \frac{f_P^2 m_P^2}{|\langle 0 | K_P^0 | K^0 \rangle|^2} \equiv X(m_q), \quad (17)$$

where  $m_P$  and  $f_P$  are the mass and the decay constant of the pseudoscalar meson consisting of two degenerate quarks of mass  $m_q$ .

Before discussing our results for  $\hat{B}_K$ , two important remarks are in order. In the calculation of  $\hat{R}^{\text{stand.}}$  we used the subtraction constants given in [10] where beside the statistical ones we also quoted the systematic uncertainties which arise from the spread of values obtained at various values of the momenta flowing through external legs of the elementary vertices (i.e., various  $a\mu$  in the RI/MOM scheme). In this paper the subtraction and renormalization constants computed in [10] are combined with bare matrix elements. The above mentioned systematic uncertainties are accounted for by computing the renormalized and subtracted matrix element for each value of  $a\mu$  separately. A second important remark is related to the accuracy of the two methods employed in this paper. From Table 1 we see that the errors in  $\hat{R}^{\text{w/o subtr.}}(a)$  and in  $\hat{R}^{\text{stand.}}(a)$  are comparable. As mentioned in the introduction the computation of the four-point correlation function needed for  $\hat{R}^{\text{w/o subtr.}}(a)$  is more demanding so that – even though one avoids making subtractions – the statistical errors of the two methods are essentially equal. The benefit of the method without subtraction is therefore not in improving the statistical quality of the results but rather in preventing the occurrence of uncontrollable systematic uncertainties that might plague the standard method due to delicate cancellations of subtractions.

### 3.2 $\hat{B}_K$

With Wilson fermions,  $\mathcal{O}(a)$  lattice artifacts can affect the chiral behavior of the matrix elements relevant to the computation of  $B_K$ . A convenient way for a clean extraction of  $B_K$  has been explained in [17] and consists in studying the dependence of the ratios  $\hat{R}^{\text{stand.}}$  and  $\hat{R}^{\text{w/o subtr.}}$  as

**Table 2.** Results for the  $\hat{B}_K$  parameter as obtained through the fit (18) for all three values of the lattice spacing and by using both strategies (standard and the one without subtractions). The values of  $a/r_0$  are taken from [15]. We also show the results of the linear extrapolation in lattice spacing to the continuum limit

$\beta$	$a/r_0$	$\hat{B}_K^{\text{stand.}}$	$\hat{B}_K^{\text{w/o subtr.}}$
6.0	0.1863	1.119(54)	1.066(39)
6.2	0.1354	1.074(49)	1.041(37)
6.4	0.1027	1.058(44)	1.017(46)
$\infty$	0	0.980(114)	0.961(103)

functions of  $X$ , namely

$$\hat{R} = \alpha + \beta X, \quad (18)$$

where the fit parameter  $\beta$  is identified as  $\hat{B}_K(a)$ , and  $\alpha$  is the parameter that measures the fitness of the chiral behavior of the ratios  $\hat{R}$ . We find that  $\alpha$  for all our lattices is consistent with zero. From such fits, at each lattice spacing, we thus obtain  $\hat{B}_K(a)$ , all of which are listed in Table 2.

In the same table we also present the results of the extrapolation to the continuum limit. That extrapolation has been made linearly since none of the operators used in (5), (10) and (12) has been improved. We see that the results of the two procedures lead to a consistent value in the continuum limit. That may also be viewed as evidence that no uncontrollable systematic uncertainty used in the standard method occurred while performing the delicate subtraction procedure. If we imposed the two methods to produce exactly the same result in the continuum limit (similar to what has been done in [18]), we would have obtained

$$\hat{B}_K = 0.969(67). \quad (19)$$

Our errors, after extrapolating to the continuum limit are quite large anyway and we do not attempt to include the quadratic term in the continuum extrapolation. The physical volume of all our lattices is about  $(1.7 \text{ fm})^3$ . By using the formulae of [19] it turns out that for the pseudoscalar mesons consisting of degenerate quarks and with mass  $m_P \gtrsim 500 \text{ MeV}$ , the finite volume effects are negligibly small. In the realistic situation, however, one of the valence quarks is the strange one (which we can work with directly on the lattice) and the other is the  $d$ -quark. That situation would necessitate the chiral extrapolation, which in the quenched theory would suffer from the (divergent) quenched chiral logarithms. To assess some uncertainty due to the degeneracy we may take the relative difference between the chiral logarithmic part known in the degenerate and non-degenerate case in full ChPT. With  $A_\chi = 1 \text{ GeV}$ , we obtain that  $\hat{B}_K$  for the kaon with non-degenerate quarks would be only 2% smaller than the one with degenerate quarks. Finally since our calculations are made in the quenched approximation, our result cannot make an impact on the world average value for  $\hat{B}_K$ , which

is actually completely dominated by the errors due to the use of a quenched approximation [20]<sup>3</sup>. It is worth mentioning that the short distance piece in the unquenched scenario would lead to  $\hat{B}_K$  larger by only 1% ÷ 2% compared to the quenched one. Such an estimate arises after replacing  $n_f = 0$  by  $n_f = 4$  in (14) and in  $\alpha_s(\mu)$ , and by using  $\Lambda_{\overline{\text{MS}}}^{(n_f=4)} = 294_{-38}^{+42} \text{ MeV}$  [21].

## 4 Conclusion

In this letter we presented the results for the renormalization group invariant bag parameter,  $\hat{B}_K$ , computed on the lattice with Wilson quarks. Besides the standard procedure, which requires a delicate subtraction of the spurious mixing with other  $\Delta S = 2$ , dimension-six, four-quark operators, we also implemented the method based on the use of a Ward identity that allows us to avoid the subtraction procedure altogether.

Our lattice data are produced in the quenched approximation at three values of the lattice spacing. At each lattice spacing we use the non-perturbatively computed renormalization and subtraction constants, presented in [10]. The conversion to the standard renormalization invariant form is made after checking that our data follow the renormalization scale dependence described by the RI/MOM anomalous dimension coefficients known to two-loops in perturbation theory. After having extrapolated to the continuum limit we obtain the physically relevant results quoted in Table 2 and (19). As our final estimate we chose to quote the results obtained using the method “without subtractions”, namely

$$\hat{B}_K = 0.96(10). \quad (20)$$

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## Appendix

In this appendix we show that the last two terms of (8) vanish in the  $SU(3)$  symmetric limit  $m_u = m_d = m_s = m$ .

We will use charge conjugation and  $\gamma_5$ -hermiticity which, on the quark propagators  $S_f(x, y; U)$  ( $f$  is the flavor and  $U$  the background gauge configuration), act in the following way:

charge conj.  $\mathcal{C}$

$$S_f(x, y; U) = \gamma_0 \gamma_2 S_f^T(y, x; U^C) \gamma_2 \gamma_0,$$

hermitian conj.  $\mathcal{H}$

$$S_f(x, y; U) = \gamma_5 S_f^\dagger(y, x; U) \gamma_5, \quad (21)$$

<sup>3</sup> A complete list of results for  $B_K$  by using other quark actions with recently updated references can be found in [20].

where the superscripts  $\text{T}$  and  $\dagger$  indicate respectively the transpose and the hermitian conjugation on color and Dirac indices.

Using these two symmetry properties it is easy to show that the trace of an arbitrary number of quark propagators and matrices  $\Gamma_i \in \{I, \gamma_5, \gamma_\mu, \gamma_\mu\gamma_5, \sigma_{\mu\nu} \equiv \frac{1}{2}[\gamma_\mu, \gamma_\nu], \tilde{\sigma}_{\mu\nu} \equiv \gamma_5\sigma_{\mu\nu}\}$ , computed on a gauge configuration  $U^C$ , is the complex conjugate of that computed on the gauge configuration  $U$ , i.e.,

$$\begin{aligned} & \text{Tr} [\Gamma_1 S_1(x_1, x_2; U) \Gamma_2 S_2(x_2, x_3; U) \dots \Gamma_n S_n(x_n, x_1; U)] \\ &= \text{Tr} [\Gamma_1 S_1(x_1, x_2; U^C) \Gamma_2 S_2(x_2, x_3; U^C) \dots \\ & \quad \times \Gamma_n S_n(x_n, x_1; U^C)]^* . \end{aligned} \quad (22)$$

This means that taking the real part of the trace corresponds to the inclusion of the charge-conjugated configuration  $U^c$  in the gauge average. Since the QCD action is symmetric under the charge conjugation, the average over  $N_{\text{conf.}} \rightarrow \infty$  will contain the average over the configuration  $U$  and its charge-conjugated one  $U^c$ .

Another property needed is easily obtained by using hermitian conjugation and reads

$$\begin{aligned} & \text{Tr} [\Gamma_1 S_1(x_1, x_2; U) \Gamma_2 S_2(x_2, x_3; U) \dots \Gamma_n S_n(x_n, x_1; U)] \\ &= [\prod_{i=1}^n \mathcal{E}(\Gamma_i)] \text{Tr} [S_n(x_1, x_n; U^C) \Gamma_n \dots \\ & \quad \times S_2(x_3, x_2; U^C) \Gamma_2 S_1(x_2, x_1; U^C) \Gamma_1], \end{aligned} \quad (23)$$

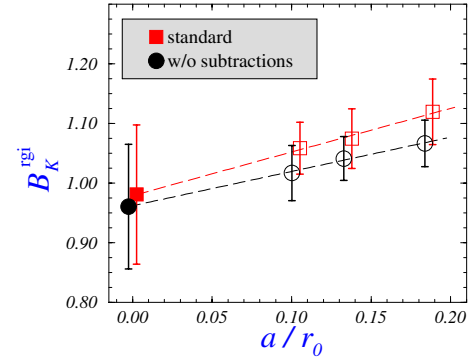
where  $\mathcal{E}(\Gamma_i) = +1$  for  $\Gamma_i \in \{I, \gamma_5, \gamma_\mu\gamma_5\}$  and  $\mathcal{E}(\Gamma_i) = -1$  for  $\Gamma_i \in \{\gamma_\mu, \sigma_{\mu\nu}, \tilde{\sigma}_{\mu\nu}\}$ .

We now analyze the correlation function of  $\mathcal{Q}_1$  between a scalar and a pseudoscalar source (since we work in the SU(3) symmetric limit we will not display the flavor indices):

$$\begin{aligned} & \frac{1}{2} \langle K_S^0(x) \mathcal{Q}_1(0) K_P^0(y) \rangle \\ &= \langle \bar{d}(x) s(x) \bar{s}(0) \gamma_\mu d(0) \bar{s}(0) \gamma_\mu \gamma_5 d(0) \\ & \quad \times \bar{d}(y) \gamma_5 s(y) \rangle \\ &= \langle \text{Tr} [S(x, 0; U) \gamma_\mu S(0, x; U)] \\ & \quad \times \text{Tr} [\gamma_\mu \gamma_5 S(0, y; U) \gamma_5 S(y, 0; U)] \\ & \quad + \text{Tr} [S(x, 0; U) \gamma_\mu \gamma_5 S(0, x; U)] \\ & \quad \times \text{Tr} [\gamma_\mu S(0, y; U) \gamma_5 S(y, 0; U)] \\ & \quad - \text{Tr} [S(x, 0; U) \gamma_\mu S(0, y; U) \gamma_5 S(y, 0; U) \\ & \quad \quad \times \gamma_\mu \gamma_5 S(0, x; U)] \\ & \quad - \text{Tr} [S(x, 0; U) \gamma_\mu \gamma_5 S(0, y; U) \gamma_5 S(y, 0; U) \\ & \quad \quad \times \gamma_\mu S(0, x; U)] \rangle_U, \end{aligned} \quad (24)$$

where  $\langle \dots \rangle_U$  denotes the average over gauge field configurations.

Using (23), we see immediately that the sum of traces in (24) is equal to the same expression computed on  $U^C$  times  $\mathcal{E}(\gamma_\mu)\mathcal{E}(\gamma_\mu\gamma_5) = -1$ . Thus, including the charge-conjugated configurations in the gauge average give identically zero for this correlator. Were we not working with degenerate  $m_s$  and  $m_u = m_d$  masses, these terms



**Fig. 3.** Extrapolation to the continuum limit. Empty symbols correspond to the results obtained at fixed lattice spacing, whereas the filled ones are the results of the linear extrapolations. The shapes of the symbols correspond to two different strategies to compute  $\hat{B}_K$ , as indicated in the legend

should be exponentially suppressed with respect to the kaon contribution in the limit of large time distances, because they correspond to the propagation of scalar states. This point can be explicitly checked by computing  $\langle \hat{K}_S^0(t_x) \hat{\mathcal{Q}}_1(0) \hat{K}_P^0(t_y) \rangle$  in the same numerical simulation for the other correlation functions appearing in (8).

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